Villarceau circles and variable-geometry toroidal coils

Eric Baird

As demonstrated by Columbia University's "Proto-CIRCUS" plasma containment device, a flexible "Villarceau torus" – an array of interlocking rigid Villarceau-circle coils with adjustable or flexible mountings – can be used to create toroidal fields with variable geometry. We review some basic results relevant to Villarceau circles and fixed and variable Villarceau coils.

doi: 10.13140/RG.2.2.29729.51047/1

1. Introduction

Devices that generate toroidal ("doughnut-shaped") fields are commonly used to accelerate and contain charged particles and or plasma, with the closed (usually circular) path allowing high-speed particles to be confined in a restricted space in an orderly manner. These fields are traditionally generated by field coils with stacked circular or toroidal windings. However, a similar effect can be produced with a torus approximated by multiple interlinked and tilted circular coils arranged as individual "Villarceau circles", [1, 2, 3, 4, 5] a configuration that has been the subject of ongoing research at the University of Columbia. [6, 7, 8, 9] This paper summarises some of the basic results regarding Villarceau circles and coils.

2. Villarceau Circles

2.1. Concept

Any point on a torus surface can be intersected by two "obvious" circles drawn on the surface: one parallel to the torus plane that circles the major axis, and a second at 90 degrees to the first (at right angles to the torus plane), which circles the torus limb. Less intuitively, the surface supports two further circles passing through the same point, tilted at (+/-) a particular angle to the plane, that varies with the torus proportions. The existence of these additional circles was documented by Yvon Villarceau (1813–1883) [1], who pointed out that any torus can be bisected to give a cross-section whose edge shows a matched pair of these circles, overlapping.

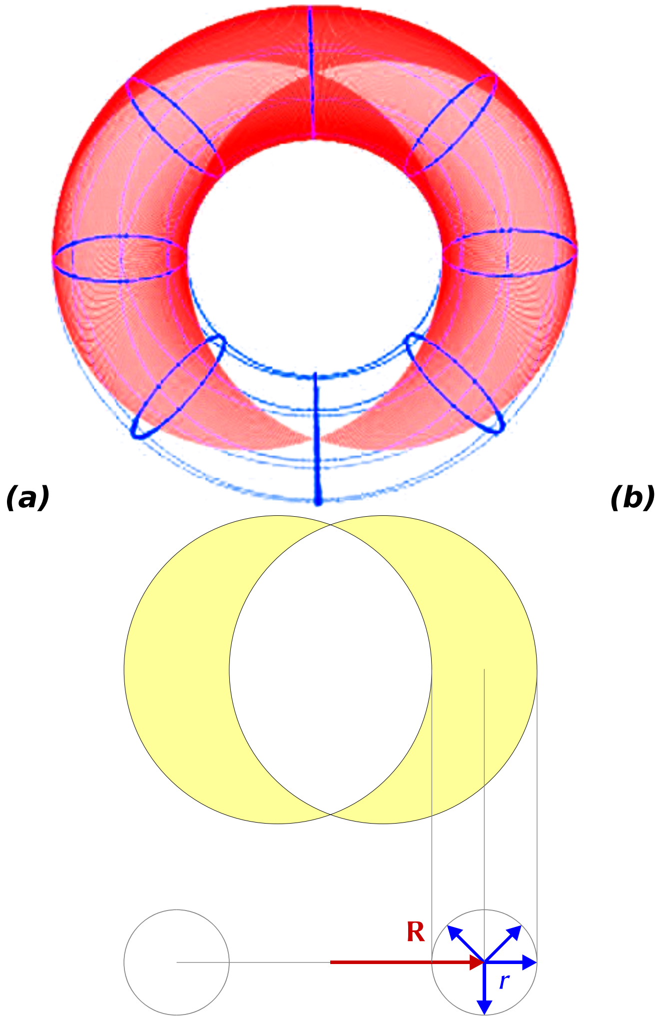

Figure 1(a): *One half of a torus, tilted at 20 degrees to the viewing plane and bisected through the plane,* ***(b):*** *the resulting Villarceau circles. The slice produces a surface with two areas that meet at a pair of points, with a gap between. The region's boundary forms a pair of intersecting circles. Some more "obvious" circular cross-sections are shown in blue.*

Every point on a conventional torus is therefore intersected by *four* circles: two representing horizontal and vertical cuts through the torus, and another two representing angled cuts through the point, in two planes, set at plus/minus the Villarceau angle.

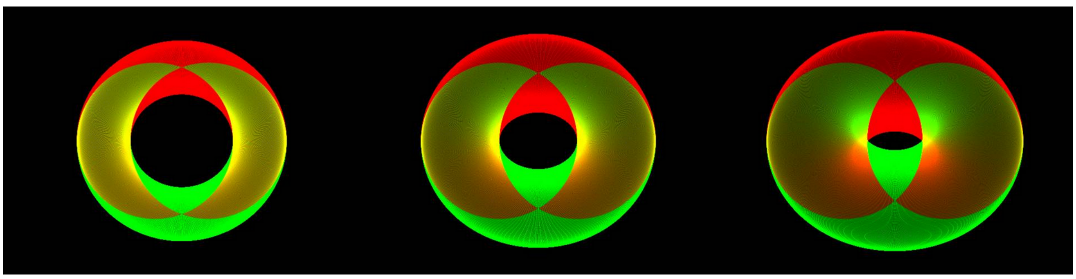

Figure 2: Generality. *Three different self-illuminated transparent toroidal surfaces, tilted at their respective Villarceau angles (viewing plane = Villarceau plane). The torus surface below the Villarceau plane*

is coloured red, that above it green. “Pure red” regions are below the plane, “pure green” regions are above the plane, and more ambiguous colours mean that we are seeing above-plane and below-plane surfaces superimposed – a region intersected by the Villarceau plane.

3. Geometry

3.1. Cut angle

If the major radius **R** describes the radius of the circle running through the central core of the torus limb, and the minor radius ***r*** is the limb’s “fatness”, then the special “Villarceau angle” **A** (away from the torus plane) is:

$$\mathbf{SIN\ A} = \boldsymbol{r}/\mathbf{R}$$

The minor radius ***r*** can have values between zero and **R**: for ***r*=0**, the torus has zero thickness and becomes a volumeless circle, and for ***r***=**R**, the size of the central throat shrinks to zero, and the inner rim of the torus shrinks to a point. For ***r***>**R**, the shape self-intersects.

3.2. Overlap

This range can be expressed as the amount of overlap between the Villarceau circle-pairs: when ***r*=0** and the torus reduces to a simple circle, the two circles on the surface of the torus necessarily coincide (figure 3, left), when ***r*=R** to give the maximally-fat torus, the circles are effectively side-by side (figure 3, right). Any conventional torus therefore gives a Villarceau-circle overlap somewhere between zero and 100%.

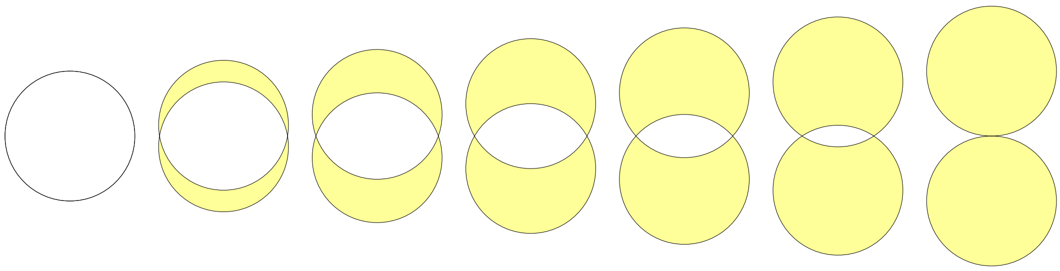

Figure 3: *Villarceau circle-pairs for a range of torus proportions from a maximally-thin torus, r/R=0 (left) to a maximally-fat torus, r/R=1 (right)*

The circle-overlap, **2R-2r**, is also the diameter of the torus throat (torus inner rim)**.**

For a given angle **A** and major radius **R**, the offset of each Villarceau circle’s centre from the common centre (required to make a circle reach the torus outer rim) is just the minor radius ***r***,

$$\boldsymbol{r} = \mathbf{R\ SIN\ A}$$

3.3. Villarceau circle radius

If we start with **r=0** and two coincident Villarceau circles, the Villarceau radius is just $\mathbf{R_V=R}$. If we then give **r** a non-zero value, the point at which a circle crosses the torus outer rim is displaced outwards by **r** … but the point at which it crosses the *inner* rim is *also* displaced, in the same direction, by **r**. The Villarceau circle radius is therefore $\mathbf{R_V=R}$ regardless of the torus' proportions.

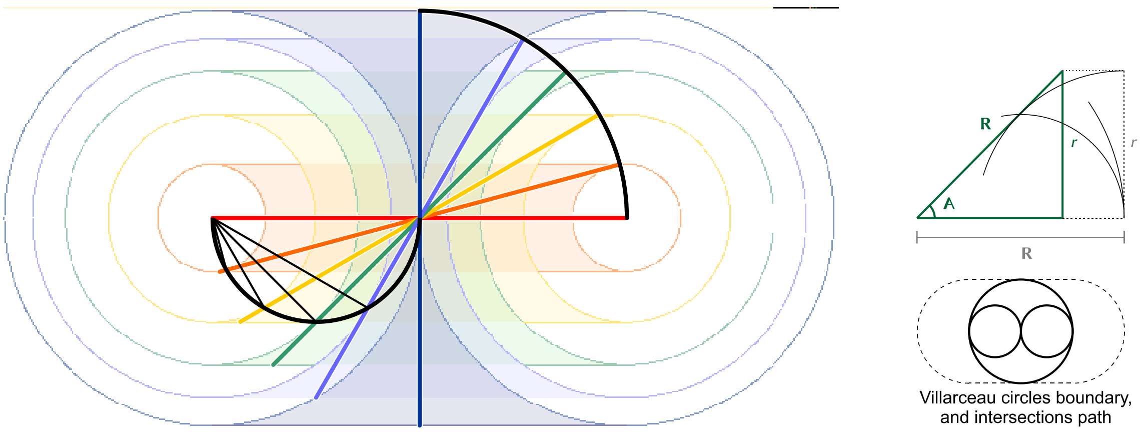

Figure 4: *Cross-section of nested tori with major radius R, and their corresponding Villarceau circle planes. The Villarceau circles all have radius R, and fit into a cylinder of radius R intersecting the tori. The Villarceau circle intersection points appear to lie on a pair of smaller circles perpendicular to the plane, of radius R/2 (figure, leftmost part).*

The "height" of the circles measured within the plane is **2R**, and the circles must also traverse the highest and lowest surfaces of the torus, giving a vertically-measured height of **2r** … so the angle **A** that the plane makes with the horizontal must be **SIN A =r/R** , as stated in section 3.1.

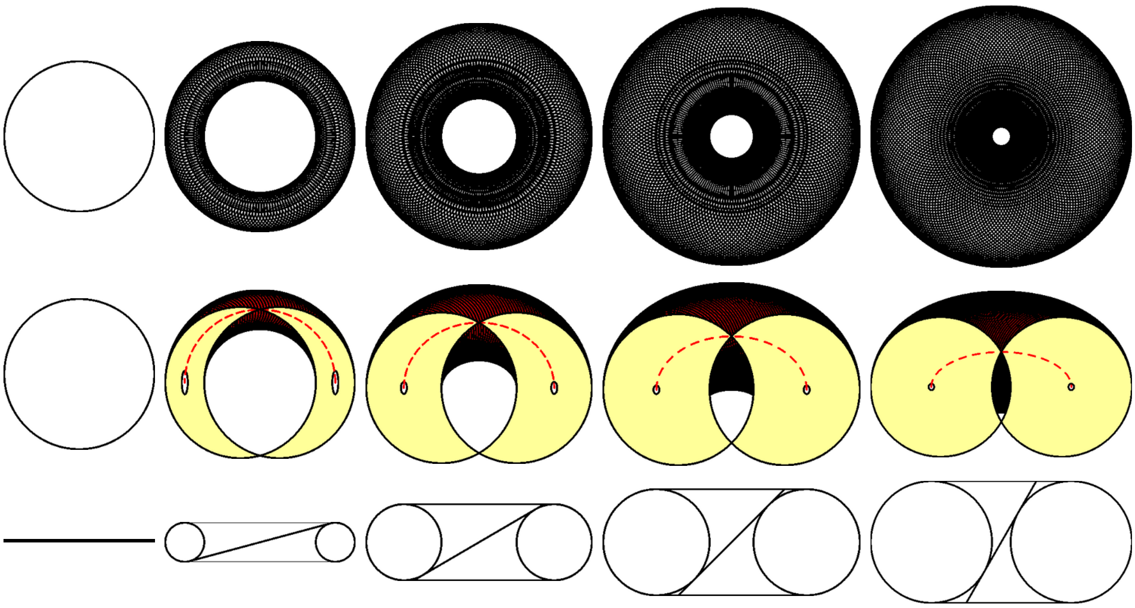

Figure 5: *Villarceau cross-sections, fixed-R, at 0, 15, 30, 45 and 60 degrees (r/R= 0, ~0.2588, 0.5, ~0.7071, ~0.8660). As the thickness of the torus limb increases from r=0 to r=R, the Villarceau circle radius R_V remains constant, R_V=R.*

4. Fixed-geometry Villarceau coils

Having familiarised ourselves with the Villarceau geometry, we can now consider the case of a single Villarceau circle on the surface of a torus:

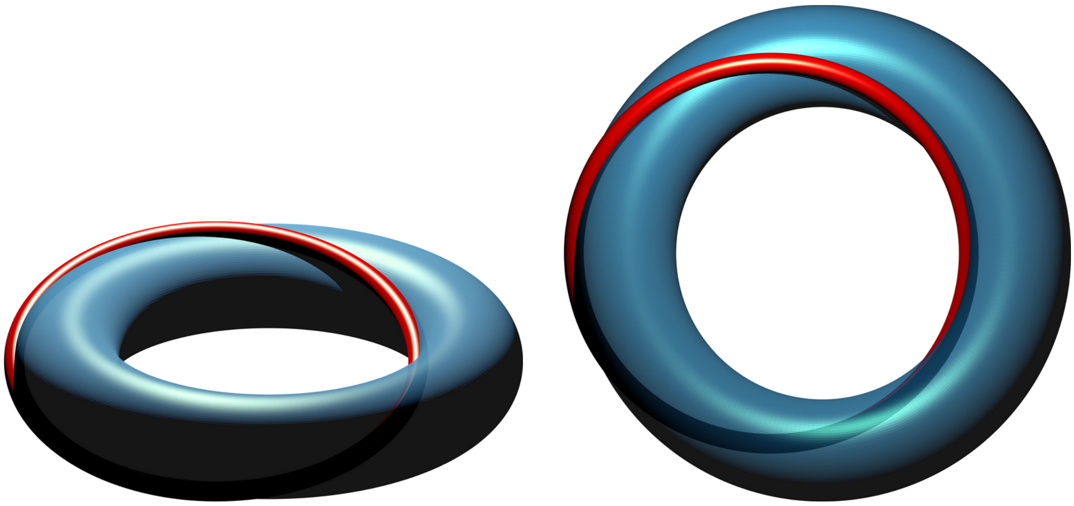

Figure 6: *A single Villarceau ring (red) embedded in the surface of a larger torus. Two interlinked rings are the minimum required to define the torus' proportions, dimensions and orientation.*

If this circle is replaced by a loop of wire, we have a "special case" of a toroidal coil winding, where the number of turns of wire around the torus limb required to traverse the entire limb and return to the starting point is "one". This would obviously be a somewhat "open" winding (!), but with two or more of these tilted rings (which could each be multi-turn circular coils) arranged around the torus, interlinked and tilted in sympathy with their immediate neighbours, we can cover the torus arbitrarily densely – the limiting factor being the width of each individual ring.

Figure 7: *A nine-ring Villarceau coil*

If our circular rings are powered independently, or additional wiring is used to connect them in series or parallel, the overall effect is similar to that produced by a single, more conventional toroidal winding.

Since identical rings can be stacked to produce "fat" or "thin" toroidal shells with the same major radius **R**, these shells can also be concentrically layered or "nested" (like Russian dolls) around a common circle of radius **R**, [5] and a "shuffling" of the arrangement of a fixed number of rings can then produce not just different sizes of shell, but single or multiple concentric shells.

5. The variable-geometry Villarceau coil

A single array of rigid rings of negligible thickness can (in theory) be used to simulate any toroidal coil with proportions ranging from **r>>~0** (rings almost horizontal) to **r>>R** (rings almost vertical). In practice, ring thickness is non-negligible and how far we can approach these extremes will be limited by the thickness of each ring and the number of rings involved.

If the array is hinged, articulated or otherwise flexible, the torus proportions can be adjusted for different situations, or even changed while operating, without changing the radius **R** of the central path within the torus.

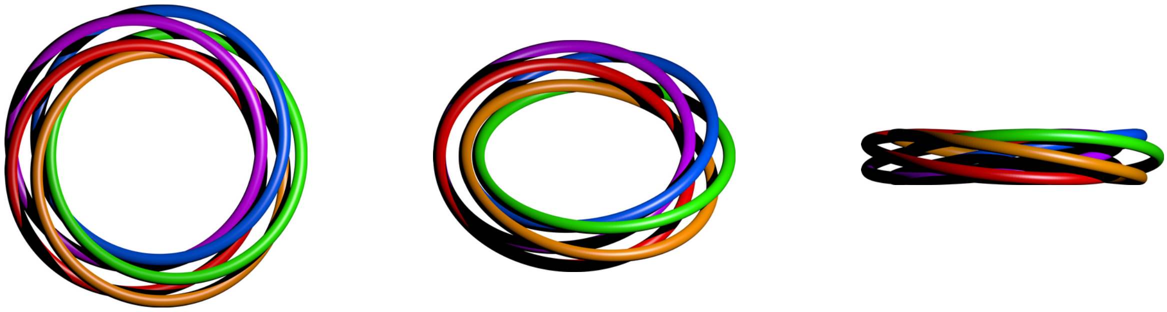

Figure 8(a): *A "tight" configuration of five rings*

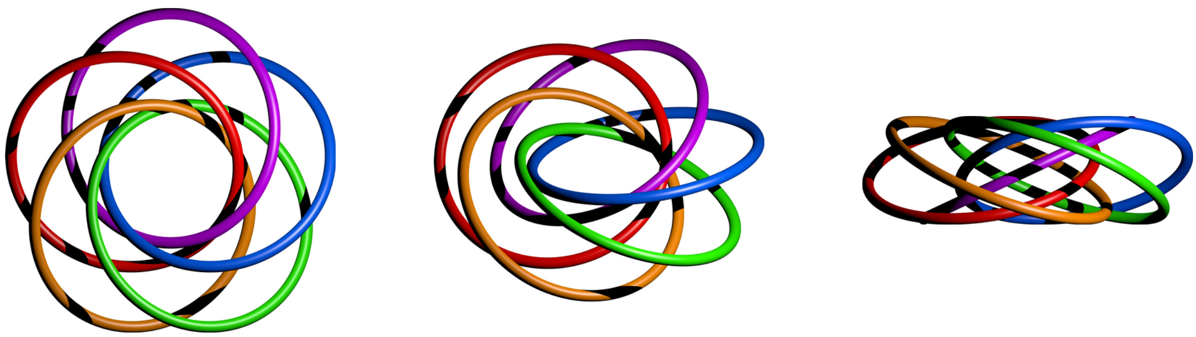

Figure 8(b): *The same five rings tilted more upright, and set a little further apart to create a fatter toroidal coil.*

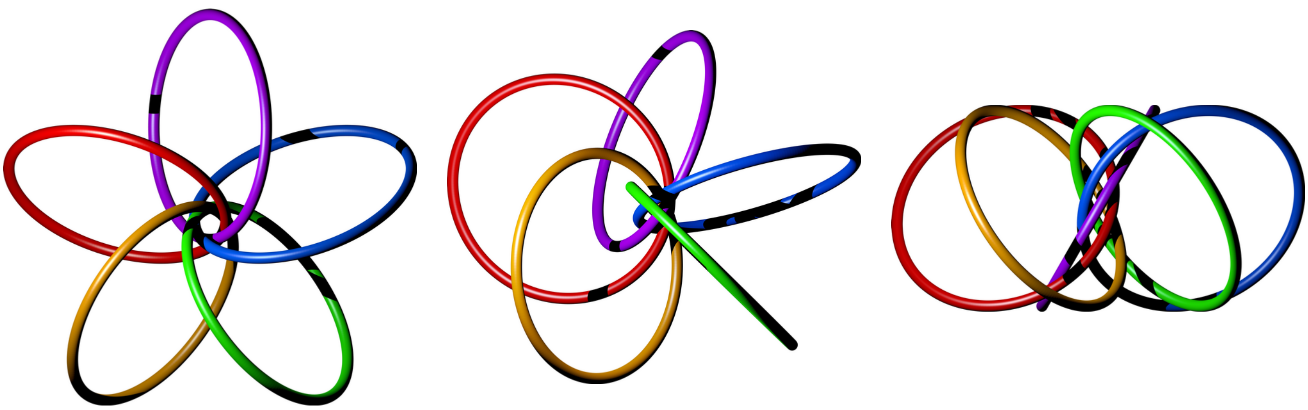

Figure 8(c): *The same five rings, adjusted again to produce an even fatter and more open toroidal coil.*

This suggests some intriguing possibilities: if the rings are spaced apart with flexible mounts, an array of "floating" rings may be able to physically adjust and adapt to slight irregularities in the field. As well as allowing the same hardware to be optimised for different tasks, there may be some advantages to having a *dynamic* array – a set of containment rings could be comparatively "open" at the start of an experiment for inspection and loading, be progressively "closed" once the experiment is underway to produce a more compact and intense field, and then allowed to relax open again to allow the removal of reaction products. The configuration may also be useful for other, future technologies requiring toroidal containment and/or particle flow.

6. Conclusions

While a "textbook-style" geometrical approach proves that any torus can be decomposed into Villarceau circles, the reverse approach says that a single size of circle can be used to reconstruct conventional tori of any proportion, giving us the concept of the variable-geometry Villarceau coil, and (if the rings are field coils) a variable-geometry field device. The simplicity of the components and the ability to alter the geometry, perhaps even in real time, suggests a possible more sophisticated alternative to tokamak-type plasma containment devices.

Much work on this subject has already been done by the teams at Columbia University, initially with the "Columbia Non-neutral Torus" project (CNT) using a single pair of interlinked rings, and then with the "Proto-CIRCUS" project. The latter team's 2014 paper shows a working desktop plasma device with six interlinked circular coils, and special mountings to allow adjustment of coil angles and spacings to produce different torus geometries.

Although some of the results given here may appear trivial (such as $\mathbf{R_V=R}$ and the Villarceau angle), they can also be surprisingly difficult to find in the standard literature. It is hoped that this paper, as an "introductory primer", will help to introduce a wider audience to this intriguing subject and its possible applications.

References

1. Yvon Villarceau, "*Extrait d'une note communiquée à M. Babinet par M. Yvon Villarceau*", Comptes-rendus de l'Académie des sciences, t. **27** (1848), page 246
   reproduced in "*Les cercles de Villarceau sur le tore*" March 2010
   http://www.bibnum.education.fr/mathematiques/geometrie/les-cercles-de-villarceau-sur-le-tore
2. "*Section du tore par un plan tangent à cette surface et passant par son centre*", Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale, Série 1 : Volume **18** (1859), pages 258-261. Online at http://archive.numdam.org/article/NAM_1859_1_18__258_1.pdf
3. Eric Baird, "*The Villarceau Coil*", blog post, 30 October 2009
   http://erkdemon.blogspot.com/2009/10/villarceau-coil.html
4. Alexandre Moatti, "*Les cercles du tore*", "Blog Sciences" post, 8 November 2009
   http://www.maths-et-physique.net/article-les-cercles-du-tore-38982808.html

5. Eric Baird, **Alt Fractals: A visual guide to fractal geometry and design** (Chocolate Tree Books, 2011) Section 45: "*Villarceau Coils and Toroidal Packings*", pages 210-213 ISBN 9780955706837
6. T.N. Todd, "*Ultra-simple stellarators*", Plasma Physics and Controlled Fusion, vol. **32**, (6) p.459 (1990) https://doi.org/10.1088/0741-3335/32/6/004
7. Thomas Sunn Pedersen and Allen H. Boozer "*Confinement of Nonneutral Plasmas on Magnetic Surfaces*" Phys. Rev. Lett. vol. **88**, 205002 (2002) https://doi.org/10.1103/PhysRevLett.88.205002
8. T. Sunn Pedersen, J. P. Kremer *et. al.*, "*Construction and Initial Operation of the Columbia Nonneutral Torus*" Fusion Science and Technology vol. **50** (3) in Fusion Science and Technology [IF: 0.58] () https://doi.org/10.13182/FST06-A1258
9. A.W. Clark, M. Doumet *et. al.*, "*Proto-CIRCUS tilted-coil tokamak–torsatron hybrid: Design and construction*" Fusion Engineering and Design vol. **89**, 2732-2737 (2014), https://doi.org/10.1016/j.fusengdes.2014.07.012